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Invariant and attracting sets of non-autonomous impulsive neutral integro-differential equations

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Abstract

This paper is concerned with a non-autonomous impulsive neutral integro-differential equation with time-varying delays. We establish a novel singular delay integro-differential inequality, which enables us to derive several sufficient criteria on the positive invariant set, global attracting set and stability. An example is given to demonstrate the efficiency of proposed results.

Keywords: Impulsive; Neutral Integro-differential Equation; Attracting Set; Stability.
MSC: 34D45, 45J05.

1 Introduction

Due to the plentiful dynamical behaviors, integro-differential equations with delays have many applications in a variety of fields such as control theory, biology, ecology, medicine, etc [1, 2]. Especially, the effects of delays on the stability of integro-differential equations have been extensively studied in the previous literature (see [3]-[9] and references cited therein).

Besides delays, impulsive effect usually exist in many evolution processes in which the states exhibit abrupt changes at certain moments, such as threshold phenomena in biology, bursting rhythm models in medicine and frequency modulated systems, etc. In recent years, the theory of impulsive integro-differential equations with delays has attracted wide attention and lots of significant results on existence, initial (boundary) value problems and stability have been reported [10]-[20]. Some results for impulsive neutral differential equations with delays have been published. For instance, in [21], the exponential stability for impulsive neutral differential equations with finite delays has been studied by using differential inequality technique. In [22, 23], some stability conditions based on Lyapunov-Krasovkii functional method have been established for impulsive neutral differential equations with finite delays. In [24], authors studied the exponential stability for impulsive neutral integro-differential equations with delays by developing a singular integro-differential inequality. However, in general, the results about impulsive neutral differential equations with delays are still scarce due to some theoretical and technical difficulties.

Additionally, it worth noting that those results in previous literature [21]-[24] have only focused on the stability of the equilibrium point for autonomous impulsive neutral differential equations with delays. However, under impulsive perturbation, the equilibrium point sometimes does not exist in many real physical systems, especially in nonlinear and non-autonomous dynamical systems. Therefore, an interesting and more general issue is to discuss the invariant set and attracting set of non-autonomous impulsive systems. Some important progress has been made in the techniques and methods for determining the invariant and attracting sets of delay differential equations [25, 26], impulsive differential equations with delays [27] and neutral differential equations [28]. Until now the corresponding problems for impulsive neutral differential (or integro-differential) equations with delays have not been considered.

Motivated by the above discussion, we will investigate the asymptotic behaviors of solutions for a non-autonomous impulsive neutral integro-differential equation with time-varying delays in this

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paper. As shown in [20, 21, 24], differential inequalities are very important tools to investigate dynamical behaviors of differential equations. We shall develop a novel singular delay integro-differential inequality in Section 3. Compared with those existing results such as (7) in [20], (8) in [21] and (16) in [24], the presented inequality (formulated by the later inequality (6)) has the following improvements.

(a) All of those key inequalities established in [20, 21, 24] are autonomous. That is to say the involved coefficients are constants. However, in this paper, the presented singular integro-differential inequality is non-autonomous, which means the coefficients are time varying.

(b) In the proposed inequality (6), the additional input term J is very novel and crucial for our studying. If $J \neq 0$, we can use the inequality to estimate the positive invariant set and global attracting set explicitly. If $J = 0$, inequality (6) can cover those inequalities in [20, 21, 24] and enable us to investigate the stability of the equilibrium point.

In Section 4, by using the transform technique similar to [21, 24], we derive some sufficient criteria on the global attracting set, positive invariant set and stability. In Section 5, an example and its simulations are given. Finally, we make some conclusions.

2 Notations and Model Description

Let $\mathbb{R}^n$ be the space of n-dimensional real column vectors and $\mathbb{R}^{m\times n}$ be the class of $m \times n$ matrices with real components. The inequality " $\leq$ " (" $>$ ") between matrices or vectors such as $A \leq B$ ($A > B$) means that each pair of corresponding elements of A and B satisfies the inequality " $\leq$ " (" $>$ "). $A \in \mathbb{R}^{m\times n}$ is called a nonnegative matrix if $A \geq 0$ and $x \in \mathbb{R}^n$ is called a positive vector if $x > 0$. x^{T} and A^{-1} denote the transpose of a vector x and the inverse of a square matrix A, respectively. I denotes the identity matrix with appropriate dimensions. $\mathcal{N} = \{1, 2, \ldots, n\}$, $\mathbb{Z}^+ = \{1, 2, \cdots\}$.

For $A \in \mathbb{R}^{m\times n}$ and function $x(t) = (x_1(t), \cdots, x_n(t))^{\mathrm{T}} \in \mathbb{R}^n$ defined on $\mathbb{R}$, we use notations

$$
\begin{gathered}
[A]^+ = (|a_{ij}|)_{m\times n}, \quad [x(t)]^+ = (|x_1(t)|, \cdots, |x_n(t)|)^{\mathrm{T}}, \\
[x(t)]_\tau = ([x_1(t)]_\tau, \ldots, [x_n(t)]_\tau)^{\mathrm{T}}, \quad [x(t)]^+_\tau = \left[[x(t)]^+\right]_\tau, \\
[x(t)]_\infty = ([x_1(t)]_\infty, \ldots, [x_n(t)]_\infty)^{\mathrm{T}}, \quad [x(t)]^+_\infty = \left[[x(t)]^+\right]_\infty, \\
[x_i(t)]_\tau = \sup_{-\tau\leq s\leq 0} x_i(t+s), \quad [x_i(t)]_\infty = \sup_{-\infty< s\leq 0} x_i(t+s), i \in \mathcal{N}.
\end{gathered}
$$

$C[X, Y]$ denotes the space of continuous mappings from the topological space X to the topological space Y.

$PC[J, \Omega] = \Big\{\psi : J \to \Omega | \psi(s)$ is continuous for all but at most countable points $s \in J$ and at these points, $\psi(s^+)$ and $\psi(s^-)$ exist, $\psi(s) = \psi(s^+)$ and $\sup\limits_{s\in J}[\psi(s)]^+ < +\infty\Big\}$. Here $J \subseteq \mathbb{R}$ is an interval and $\Omega \subseteq \mathbb{R}^n$, $\psi(s^+)$ and $\psi(s^-)$ denote the right-hand and left-hand limits of the function $\psi(s)$, respectively.

$PC^1[J, \Omega] = \Big\{\psi : J \to \Omega | \psi(s)$ is continuously differentiable for all but at most countable points $s \in J$ and at these points, $\psi(s^+), \psi(s^-), \psi'(s^+)$ and $\psi'(s^-)$ exist, $\psi(s) = \psi(s^+), \psi'(s) := \psi'(s^+)$ and $\sup\limits_{s\in J}[\psi(s)]^+ < +\infty, \sup\limits_{s\in J}[\psi'(s)]^+ < +\infty\Big\}$. $\psi'(s)$ denotes the derivative of $\psi(s)$.

$L(\sigma_0) = \Big\{\psi : [0, +\infty) \to \mathbb{R} | \psi(s)$ is piecewise continuous and satisfies $\int_0^{+\infty} |\psi(s)|e^{\sigma_0 s}ds < +\infty$ for some constant $\sigma_0 > 0\Big\}$.

For $\phi \in PC := PC[(-\infty, 0], \mathbb{R}^n]$, $\psi \in PC^1 := PC^1[(-\infty, 0], \mathbb{R}^n]$ and $x \in \mathbb{R}^n$, we use the following norms

$$
||\phi||_\infty = \max_{i\in\mathcal{N}} \left\{[\phi_i(s)]^+_\infty\right\}, \quad ||\psi||_{1\infty} = \max_{i\in\mathcal{N}} \left\{[\psi_i(s)]^+_\infty, [\psi'_i(s)]^+_\infty\right\}, \quad ||x|| = \max_{i\in\mathcal{N}}\{|x_i|\}.
$$

Consider a non-autonomous impulsive neutral integro-differential equation with time-varying delays

$$\begin{cases} x_i'(t) = \beta(t)\Big[- d_i x_i(t) + \sum\limits_{j=1}^{n} (a_{ij} f_{ij}(x_j(t)) + b_{ij} g_{ij}(x_j(t - \tau_{ij}(t))) + c_{ij} h_{ij}(x_j'(t - r_{ij}(t))) \\ \qquad\qquad + \int\limits_{-\infty}^{t} k_{ij}(t-s) p_{ij}(x_j(s)) ds) + l_i(t)\Big], \quad t \geq t_0, t \neq t_k, \\ x_i(t) = x_i(t^+) = I_{ik}(x_1(t^-), \ldots, x_n(t^-)), \qquad t = t_k, \end{cases} \tag{1}$$

with the initial condition

$$x_i(t_0 + s) = \phi_i(s), \quad -\infty < s \leq 0, \tag{2}$$

where

(H_1) f_{ij}, g_{ij}, h_{ij}, p_{ij}, τ_{ij}, r_{ij}, l_i and $\beta \in C[\mathbb{R}, \mathbb{R}]$; $0 < \beta(t) \leq \hat{\beta}$ and $0 \leq \{\tau_{ij}(t), r_{ij}(t)\} \leq \tau$ for all $t \geq t_0$ and $i, j \in \mathcal{N}$; $\lim\limits_{t \to +\infty} \int\limits_0^t \beta(u) du = +\infty$.

(H_2) $\hat{\beta} > 0$, $\tau \geq 0$, $d_i > 0$, a_{ij}, b_{ij} and c_{ij} are constants.

(H_3) For all $k \in \mathbb{Z}^+$, the jump functions $I_k = (I_{1k}, \ldots, I_{nk})^{\mathrm{T}} \in C[\mathbb{R}^n, \mathbb{R}^n]$ and the fixed impulsive moments satisfy $t_k < t_{k+1}$, $\lim\limits_{k \to +\infty} t_k = +\infty$.

(H_4) The initial condition $\phi = (\phi_1(s), \ldots, \phi_n(s))^{\mathrm{T}} \in PC^1$.

Remark 2.1. Clearly, (1) is a general form of many popular systems studied extensively in [20]-[22], [24]-[28].

For any initial condition $\phi \in PC^1$, we always assume that (1) has a solution denoted by $x(t, t_0, \phi)$ or $x_t(t_0, \phi)$ (simply $x(t)$ or x_t if no confusion occurs), where $x_t(t_0, \phi) = x(t + s, t_0, \phi), -\infty < s \leq 0$. We know $x(t)$ is continuously differentiable for $t \geq t_0$ and $t \neq t_k$. Moreover $x(t)$ has discontinuities of the first type at the fixed impulsive moments t_k. Namely, $x_t \in PC^1$. For convenience, we denote $x'(t_k) = x'(t_k^+)$.

Let $x'(t) = y(t)$. The model (1) be transformed to an $2n$-dimensional non-autonomous singular impulsive integro-differential equation as follows

$$\begin{cases} x_i'(t) = \beta(t)\Big[- d_i x_i(t) + \sum\limits_{j=1}^{n} (a_{ij} f_{ij}(x_j(t)) + b_{ij} g_{ij}(x_j(t - \tau_{ij}(t))) + c_{ij} h_{ij}(y_j(t - r_{ij}(t))) \\ \qquad\qquad + \int\limits_{-\infty}^{t} k_{ij}(t-s) p_{ij}(x_j(s)) ds) + l_i(t)\Big], \quad t \geq t_0, t \neq t_k, \\ y_i(t) = \beta(t)\Big[- d_i x_i(t) + \sum\limits_{j=1}^{n} (a_{ij} f_{ij}(x_j(t)) + b_{ij} g_{ij}(x_j(t - \tau_{ij}(t))) + c_{ij} h_{ij}(y_j(t - r_{ij}(t))) \\ \qquad\qquad + \int\limits_{-\infty}^{t} k_{ij}(t-s) p_{ij}(x_j(s)) ds) + l_i(t)\Big], \quad t \geq t_0, t \neq t_k, \\ x_i(t) = x_i(t^+) = I_{ik}(x_1(t^-), \ldots, x_n(t^-)), \quad t = t_k, \\ y_i(t) = y_i(t^+) = x_i'(t^+), \qquad t = t_k, \end{cases} \tag{3}$$

with the initial condition

$$\begin{cases} x_i(t_0 + s) = \phi_i(s), \quad -\infty < s \leq 0, \\ y_i(t_0 + s) = \phi_i'(s), \quad -\infty < s \leq 0. \end{cases} \tag{4}$$

Remark 2.2. Recalling the definition of PC^1 and the properties of derivative function, $x_t \in PC^1$ implies that $y(t)$ has discontinuities of the first type at the fixed impulsive moments t_k and $y(t)$ is continuous on $[t_{k-1}, t_k)$ for $k \in \mathbb{Z}^+$. Therefore, studying the asymptotic behaviors of (1) in PC^1 is equivalent to those for (3) in $PC[(-\infty, 0], \mathbb{R}^{2n}]$.

Some definitions and lemma will be employed in this paper.

Definition 2.1. A set $\mathcal{A} \subset PC^1$ is called a positive invariant set of (1), if for any initial condition $\phi \in \mathcal{A}$, the solution $x_t(t_0, \phi) \in \mathcal{A}$ for $t \geq t_0$.

Definition 2.2. A set $\mathcal{B} \subset PC^1$ is called an attracting set of (1), if $\mathcal{B}$ possesses an open neighborhood $\mathcal{U}$, such that for any initial condition $\phi \in \mathcal{U}$, the solution $x(t, t_0, \phi)$ satisfies

$$\lim_{t \to +\infty} \inf_{\psi \in \mathcal{B}} \operatorname{dist}\left(x(t, t_0, \phi), [\psi]_{\infty}^{+}\right) = 0,$$

where $\mathrm{dist}(x, y)$ denotes the distance of x to y in $\mathbb{R}^n$. Particularly, if $\mathcal{U} = PC^1$, then $\mathcal{B}$ is called a global attracting set of (1).

Definition 2.3. The zero solution of (1) is called to be globally asymptotically stable in PC^1, if for any initial condition $\phi \in PC^1$, the solution $x(t, t_0, \phi)$ satisfies

$$\lim_{t\to+\infty} ||x(t, t_0, \phi)|| = 0.$$

Definition 2.4. The zero solution of (1) is called to be globally exponentially stable in PC^1, if there exist positive constants α and λ, such that for any initial condition $\phi \in PC^1$, the solution $x(t, t_0, \phi)$ satisfies

$$||x(t, t_0, \phi)|| \le \alpha||\phi||_{1\infty} e^{-\lambda(t-t_0)}, \quad t \ge t_0.$$

Definition 2.5.[20] For $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n} \in \mathbb{R}^{m\times n}$, define $A \circ B$ as follows

$$A \circ B := (a_{ij}b_{ij})_{m\times n}.$$

$A \circ B$ is called the Hadamard product or Schur product of A and B.

Definition 2.6.[29] A matrix $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ is called an $\mathcal{M}$-matrix if A has non-positive off-diagonal elements (i.e., $a_{ij} \le 0$ for $i \ne j$), and one of the following conditions holds:
(i) there exists a positive vector z such that $Az > 0$;
(ii) A^{-1} exists and $A^{-1} \ge 0$.

For an $\mathcal{M}$-matrix A, we define

$$\Omega_{\mathcal{M}}(A) = \{z \in \mathbb{R}^n | Az > 0, z > 0\}. \tag{5}$$

Obviously, Definition 2.6 leads to the following lemma.

Lemma 2.1.[21] If A is an $\mathcal{M}$-matrix, then $\Omega_{\mathcal{M}}(A) \subset \mathbb{R}^n$ is a nonempty cone without conical surface.

3 Singular Integro-differential Inequality

In what follows, we shall develop a novel non-autonomous singular delay integro-differential inequality, which is a useful tool to study impulsive delay differential equations.

Theorem 3.1. Assume $u \in C[[t_0, b), \mathbb{R}^r]$ satisfies the non-autonomous singular delay integro-differential inequality

$$\Lambda D^+u(t) \le \beta(t) \left[Pu(t) + Q[u(t)]_\tau + \int_0^{+\infty} \Psi(s)u(t-s)ds + J \right], \quad t \in [t_0, b), \tag{6}$$

with initial condition $u_{t_0} \in PC[(-\infty, 0], \mathbb{R}^r]$. Let
(C_1) $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_r\}$, where $\lambda_i > 0$ for $i \in \mathcal{N}_1^* \subseteq \mathcal{N}^* := \{1, 2, \ldots, r\}$ and $\lambda_i = 0$ for $i \in \mathcal{N}_2^* := \mathcal{N}^* - \mathcal{N}_1^*$;
(C_2) $P = (p_{ij})_{r\times r}$ with $p_{ij} \ge 0$ for $i \ne j$; $Q = (q_{ij})_{r\times r} \ge 0$; $\Psi(s) = (\psi_{ij}(s))_{r\times r} \ge 0$ with $\psi_{ij} \in L(\sigma_0)$ for $i, j \in \mathcal{N}^*$ and $J = (J_1, \ldots, J_r)^{\mathrm{T}} \ge 0$;
(C_3) there exist a positive vector $z \in \mathbb{R}^r$ and a positive constant σ such that

$$\left[\sigma\Lambda + P + Qe^{\sigma\hat{\beta}\tau} + \int_0^{+\infty} \Psi(s)e^{\sigma\hat{\beta}s}ds \right] z < 0. \tag{7}$$

If the initial condition

$$u(t) \le \kappa z e^{-\sigma \int_{t_0}^{t} \beta(u)du} + D^{-1}J, \quad t \in (-\infty, t_0], \tag{8}$$

then

$$u(t) \le \kappa z e^{-\sigma \int_{t_0}^{t} \beta(u)du} + D^{-1}J, \quad t \in [t_0, b), \tag{9}$$

where $\kappa \geq 0$ is a constant and $D = -\left(P + Q + \int\limits_0^{+\infty} \Psi(s)ds\right)$.

Proof. Recalling the definition of $L(\sigma_0)$ and $\psi_{ij} \in L(\sigma_0)$, we know $\int\limits_0^{+\infty} \psi_{ij}(s)ds < +\infty$ for $i, j \in \mathcal{N}^*$. Then

$$D = -\left(P + Q + \int_0^{+\infty} \Psi(s)ds\right) \in \mathbb{R}^{r\times r}$$

is well defined. Moreover, condition (C_2) shows D has non-positive off-diagonal elements.

Denote $z := (z_1, \cdots, z_r)^{\mathrm{T}}$. We rewrite (7) as

$$\sum_{j=1}^{r}\left(p_{ij} + q_{ij}e^{\sigma\hat{\beta}\tau} + \int_0^{+\infty} \psi_{ij}(s)e^{\sigma\hat{\beta}s}ds\right) z_j < -\sigma\lambda_i z_i, \quad i \in \mathcal{N}^*, \tag{10}$$

which implies for any $i \in \mathcal{N}^*$

$$\sum_{j=1}^{r}\left(p_{ij} + q_{ij} + \int_0^{+\infty} \psi_{ij}(s)ds\right) z_j \leq \sum_{j=1}^{r}\left(p_{ij} + q_{ij}e^{\sigma\hat{\beta}\tau} + \int_0^{+\infty} \psi_{ij}(s)e^{\sigma\hat{\beta}s}ds\right) z_j < 0.$$

That is

$$\left(P + Q + \int_0^{+\infty} \Psi(s)ds\right) z < 0 \text{ or } Dz > 0. \tag{11}$$

Consequently, by Definition 2.6, it is easy to deduce D is an $\mathcal{M}$-matrix, and D^{-1} exists with $D^{-1} \geq 0$. For simplicity, we denote

$$T = D^{-1}J := (T_1, \ldots, T_r)^{\mathrm{T}}.$$

Of course, we can see $T \geq 0$ and

$$\sum_{j=1}^{r}\left(p_{ij} + q_{ij} + \int_0^{+\infty} \psi_{ij}(s)ds\right) T_j + J_i = 0, \quad i \in \mathcal{N}^*. \tag{12}$$

Under assumption (8), we claim that for any small enough $\epsilon > 0$,

$$u_i(t) \leq (\kappa + \epsilon)z_i e^{-\sigma\int_{t_0}^{t}\beta(u)du} + T_i := v_i(t), \quad t \in [t_0, b), i \in \mathcal{N}^*. \tag{13}$$

Let us prove claim (13) by contradiction. Define

$$\mathcal{M}^* = \{i \in \mathcal{N}^* | u_i(t) > v_i(t) \text{ for some } t \in [t_0, b)\},$$

$$t_i^* = \inf\{t \in [t_0, b) | u_i(t) > v_i(t), i \in \mathcal{M}^*\}.$$

If claim (13) is false, then $\mathcal{M}^*$ is certainly a nonempty set and there must be an integer $m \in \mathcal{M}^* \subseteq \mathcal{N}^*$ such that $t_m^* = \min\limits_{i\in\mathcal{M}^*}\{t_i^*\} \in [t_0, b)$.

Case 1: If $m \in \mathcal{N}_1^*$, then by notations of $\mathcal{M}^*$, t_i^* and assumption (8) we conclude

$$u_m(t_m^*) = v_m(t_m^*), \quad u_i(t) \leq v_i(t) \text{ for all } t \in (-\infty, t_m^*], i \in \mathcal{N}^*, \tag{14}$$

and

$$D^+u_m(t_m^*) \geq v_m'(t_m^*). \tag{15}$$

On the other hand, it follows from (6) and (14) that

$$\begin{aligned}
\lambda_m D^+u(t_m^*) &\leq \beta(t_m^*)\left[\sum_{j=1}^r\left(p_{mj}u_j(t_m^*)+q_{mj}[u_j(t_m^*)]_\tau+\int_0^{+\infty}\psi_{mj}(s)u_j(t_m^*-s)ds\right)+J_m\right]\\
&\leq \beta(t_m^*)\left[\sum_{j=1}^r p_{mj}\left((\kappa+\epsilon)z_je^{-\sigma\int_{t_0}^{t_m^*}\beta(u)du}+T_j\right)\right.\\
&\quad+\sum_{j=1}^r q_{mj}\left((\kappa+\epsilon)z_je^{-\sigma\int_{t_0}^{t_m^*-\tau}\beta(u)du}+T_j\right)\\
&\quad\left.+\sum_{j=1}^r\int_0^{+\infty}\psi_{mj}(s)\left((\kappa+\epsilon)z_je^{-\sigma\int_{t_0}^{t_m^*-s}\beta(u)du}+T_j\right)ds+J_m\right]\\
&\leq \beta(t_m^*)(\kappa+\epsilon)e^{-\sigma\int_{t_0}^{t_m^*}\beta(u)du}\sum_{j=1}^r\left(p_{mj}+q_{mj}e^{\sigma\hat{\beta}\tau}+\int_0^{+\infty}\psi_{mj}(s)e^{\sigma\hat{\beta}s}ds\right)z_j\\
&\quad+\beta(t_m^*)\left[\sum_{j=1}^r\left(p_{mj}+q_{mj}+\int_0^{+\infty}\psi_{mj}(s)ds\right)T_j+J_m\right]. \qquad (16)
\end{aligned}$$

Using (10), (12) for $i=m$ and $\lambda_m>0$, inequality (16) reduces to

$$D^+u(t_m^*)<\beta(t_m^*)(\kappa+\epsilon)(-\sigma z_m)e^{-\sigma\int_{t_0}^{t_m^*}\beta(u)du}=v_m'(t_m^*),$$

which contradicts (15). So, we conclude $m\notin\mathcal{N}_1^*$.
Case 2: If $m\in\mathcal{N}_2^*$, then by recalling the notations of $\mathcal{M}^*$, t_m^* and noting (8), we derive

$$u_m(t_m^*)=v_m(t_m^*) \text{ and } u_i(t)\leq v_i(t) \text{ for all } t\in(-\infty,t_m^*], i\in\mathcal{N}^*. \qquad (17)$$

From (10), (12) for $i=m$ and $\lambda_m=0$, inequality (6) implies

$$\begin{aligned}
0 &\leq \beta(t_m^*)\left[\sum_{j=1}^r p_{mj}\left((\kappa+\epsilon)z_je^{-\sigma\int_{t_0}^{t_m^*}\beta(u)du}+T_j\right)\right.\\
&\quad+\sum_{j=1}^r q_{mj}\left((\kappa+\epsilon)z_je^{-\sigma\int_{t_0}^{t_m^*-\tau}\beta(u)du}+T_j\right)\\
&\quad\left.+\sum_{j=1}^r\int_0^{+\infty}\psi_{mj}(s)\left((\kappa+\epsilon)z_je^{-\sigma\int_{t_0}^{t_m^*-s}\beta(u)du}+T_j\right)ds+J_m\right]\\
&\leq \beta(t_m^*)(\kappa+\epsilon)e^{-\sigma\int_{t_0}^{t_m^*}\beta(u)du}\sum_{j=1}^r\left(p_{mj}+q_{mj}e^{\sigma\hat{\beta}\tau}+\int_0^{+\infty}\psi_{mj}(s)e^{\sigma\hat{\beta}s}ds\right)z_j\\
&\quad+\beta(t_m^*)\left[\sum_{j=1}^r\left(p_{mj}+q_{mj}+\int_0^{+\infty}\psi_{mj}(s)ds\right)T_j+J_m\right]\\
&< \beta(t_m^*)(\kappa+\epsilon)e^{-\sigma\int_{t_0}^{t_m^*}\beta(u)du}(-\sigma\lambda_m z_m)=0.
\end{aligned}$$

This is a contradiction, which means $m \notin \mathcal{N}_2^*$.

Hence, $\mathcal{M}^*$ can only be the empty set, which indicates claim (13) is true. Letting $\epsilon \to 0^+$, we see

$$u_i(t) \leq \kappa z_i e^{-\sigma \int_{t_0}^{t} \beta(u)du} + T_i, \quad t \in [t_0, b), i \in \mathcal{N}^*.$$

The proof is completed .

Remark 3.1. Suppose $\mathcal{N}_1^* = \mathcal{N}^*$, $\Lambda = I$, $Q = 0$, $J = 0$ and $\beta(t) \equiv 1$ for $t \in [t_0, b)$. Inequality (6) reduces to the basic inequality (7) in [20] and we can derive the Theorem 1 in [20] as a special case of the present Theorem 3.1.

Remark 3.2. Let $\beta(t) \equiv 1$ for $t \in [t_0, b)$, $p_{ij} = 0$ for $i \in \mathcal{N}^*$, $j \in \mathcal{N}_2^*$, $\psi_{ij}(s) \equiv 0$ for $s \in \mathbb{R}$, $i, j \in \mathcal{N}^*$ and $J = 0$. We can easily observe the key inequality (8) and the main result (Theorem 3.1) in [21] follow from inequality (6) and Theorem 3.1 in present paper, respectively.

Remark 3.3. The inequality (16) in [24] is a special case of (6) with $\beta(t) \equiv 1$ for $t \in [t_0, b)$, $p_{ij} = 0$ for $i \in \mathcal{N}^*$, $j \in \mathcal{N}_2^*$ and $J = 0$. That is to say our Theorem 3.1 covers the Theorem 3.1 in [24].

Remark 3.4. The basic Lemma 1 in [27] is a special case of the present Theorem 3.1 in which $\mathcal{N}_1^* = \mathcal{N}^*$, $\Lambda = I$, $\beta(t) \equiv 1$ for $t \in [t_0, b)$ and $\psi_{ij}(s) \equiv 0$ for $s \in \mathbb{R}$, $i, j \in \mathcal{N}^*$.

4 Attracting Set and Invariant Set

In this section, we will present the main results for the global attracting set, positive invariant set and stability of (3) by using the improved non-autonomous singular delay integro-differential inequality in Section 3.

For convenience, we denote $u(t) := \left(x^{\mathrm{T}}(t), y^{\mathrm{T}}(t)\right)^{\mathrm{T}}$ by the solution of (3) with any initial condition $\bar{\phi} := \left(\phi^{\mathrm{T}}, (\phi')^{\mathrm{T}}\right)^{\mathrm{T}}$. Let $z_x := (z_1, \cdots, z_n)^{\mathrm{T}}$ and $z_y := (z_{n+1}, \cdots, z_{2n})^{\mathrm{T}}$, for any $z \in \mathbb{R}^{2n}$.

The following assumptions imposed on (3) are needed in later discussion.

(A_1) There exist nonnegative constants u_{ij}, v_{ij}, w_{ij}, γ_{ij} and L_i such that

$$|f_{ij}(s)| \leq u_{ij}|s|, \quad |g_{ij}(s)| \leq v_{ij}|s|, \quad |h_{ij}(s)| \leq w_{ij}|s|, \quad |p_{ij}(s)| \leq \gamma_{ij}|s|, \quad |l_i(s)| \leq L_i,$$

for all $s \in \mathbb{R}$ and $i, j \in \mathcal{N}$.

(A_2) There exist a positive vector $\bar{z} \in \mathbb{R}^{2n}$ and a positive constant σ such that

$$\left[\sigma\bar{\Lambda} + \bar{P} + \bar{Q}e^{\sigma\hat{\beta}\tau} + \int_0^{+\infty} \bar{K}(s)e^{\sigma\hat{\beta}s}ds\right]\bar{z} < 0, \tag{18}$$

where

$$\bar{\Lambda} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \bar{P} = \begin{pmatrix} -D_0 + [A \circ U]^+ & 0 \\ D_0 + [A \circ U]^+ & -\frac{1}{\beta}I \end{pmatrix}, \tag{19}$$

$$\bar{Q} = \begin{pmatrix} [B \circ V]^+ & [C \circ W]^+ \\ [B \circ V]^+ & [C \circ W]^+ \end{pmatrix}, \bar{K}(s) = \begin{pmatrix} [K(s) \circ \Gamma]^+ & 0 \\ [K(s) \circ \Gamma]^+ & 0 \end{pmatrix}, \tag{20}$$

with

$$D_0 = \mathrm{diag}\{d_1, \ldots, d_n\}, A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n}, C = (c_{ij})_{n\times n}, U = (u_{ij})_{n\times n}, \tag{21}$$

$$V = (v_{ij})_{n\times n}, W = (w_{ij})_{n\times n}, \Gamma = (\gamma_{ij})_{n\times n}, K(s) = (k_{ij}(s))_{n\times n}, k_{ij} \in L(\sigma_0) \text{ for } i, j \in \mathcal{N}. \tag{22}$$

(A_3) There exist nonnegative matrices $R_k = (r_{ij}^k)_{n\times n}$ such that

$$[I_k(x)]^+ \leq R_k[x]^+,$$

for all $x \in \mathbb{R}^n$, $k \in \mathbb{Z}^+$.
(A_4) There exist constants $\zeta_k \geq 1$ and $\zeta \geq 0$, such that

$$\begin{pmatrix} R_k & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \bar{z}_x \\ \bar{z}_y \end{pmatrix} \leq \zeta_k \begin{pmatrix} \bar{z}_x \\ \bar{z}_y \end{pmatrix}, \tag{23}$$

and

$$\frac{\ln \zeta_k}{\int\limits_{t_{k-1}}^{t_k} \beta(u)du} \leq \zeta < \sigma, \tag{24}$$

for all $k \in \mathbb{Z}^+$, $\bar{z}$ and σ determined by (18).
(A_5) There exist constants $\nu_k \geq 1$ and $\nu \geq 0$, such that

$$\begin{pmatrix} R_k & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \bar{T}_x \\ \bar{T}_y \end{pmatrix} \leq \nu_k \begin{pmatrix} \bar{T}_x \\ \bar{T}_y \end{pmatrix}, \tag{25}$$

and

$$\sum_{k=1}^{+\infty} \ln \nu_k \leq \nu, \tag{26}$$

for all $k \in \mathbb{Z}^+$, where

$$\bar{T} = \bar{D}^{-1}\bar{L} := \left(\bar{T}_1, \cdots, \bar{T}_{2n}\right)^{\mathrm{T}}, \bar{D} = -\left(\bar{P} + \bar{Q} + \int\limits_0^{+\infty} \bar{K}(s)ds\right), \bar{L} = (L_1, \cdots, L_n, L_1, \cdots, L_n)^{\mathrm{T}}. \tag{27}$$

Theorem 4.1. Assume (A_1)-(A_5) hold. Then

$$\mathcal{B} = \left\{\bar{\phi} \in PC[(-\infty, 0], \mathbb{R}^{2n}] \middle| [\bar{\phi}]^+_\infty \leq e^\nu \bar{T}\right\}$$

is a global attracting set of (3).
Proof. At first, we claim $\bar{T}$ is well defined and $\bar{T} \geq 0$. In fact, from the definition of $L(\sigma_0)$ and condition (22), we know clearly $0 \in L(\sigma_0)$ and $\gamma_{ij}k_{ij} \in L(\sigma_0)$ for $i, j \in \mathcal{N}$. So, $\int\limits_0^{+\infty} \bar{K}(s)ds < +\infty$ and $\bar{D} = -\left(\bar{P} + \bar{Q} + \int\limits_0^{+\infty} \bar{K}(s)ds\right) \in \mathbb{R}^{2n\times 2n}$ is well defined. On the other hand, conditions (19) and (20) also show $\bar{D}$ has non-positive off-diagonal elements. By the argument similar to assertion (11), it follows easily from inequality (18) that $\bar{D}$ is an $\mathcal{M}$-matrix. That is, $\bar{D}^{-1}$ exists and $\bar{D}^{-1} \geq 0$. Hence, by condition (27), $\bar{T}$ is well defined and $\bar{T} \geq 0$.

For any $i \in \mathcal{N}$ and $t \in [t_{k-1}, t_k)$, calculating the upper right derivative $D^+|x_i(t)|$ along the solution of (3) can give

$$\begin{aligned} D^+|x_i(t)| &\leq \beta(t)\Big[- d_i|x_i(t)| + \sum_{j=1}^n \Big(|a_{ij}f_{ij}(x_j(t))| + |b_{ij}g_{ij}(x_j(t - \tau_{ij}(t)))| \\ &\quad + |c_{ij}h_{ij}(y_j(t - r_{ij}(t)))| + \int\limits_{-\infty}^t |k_{ij}(t-s)||p_{ij}(x_j(s))|ds\Big) + |l_i(t)|\Big] \\ &\leq \beta(t)\Big[- d_i|x_i(t)| + \sum_{j=1}^n \Big(|a_{ij}u_{ij}||x_j(t)| + |b_{ij}v_{ij}||x_j(t - \tau_{ij}(t))| \\ &\quad + |c_{ij}w_{ij}||y_j(t - r_{ij}(t))| + \int\limits_{-\infty}^t |k_{ij}(t-s)||\gamma_{ij}||x_j(s)|ds\Big) + L_i\Big]. \end{aligned}$$

Thus, together with conditions (21) and (22), we have a vector form as follows

$$\begin{aligned} D^+[x(t)]^+ &\leq \beta(t)\Big(- D_0[x(t)]^+ + [A\circ U]^+[x(t)]^+ + [B\circ V]^+[x(t)]^+_\tau + [C\circ W]^+[y(t)]^+_\tau \\ &+ \int_0^{+\infty} [K(s)\circ\Gamma]^+[x(t-s)]^+ds + \bar{L}_x\Big), \quad t\in[t_{k-1},t_k), k\in\mathbb{Z}^+. \end{aligned} \tag{28}$$

Meanwhile, the second equation in (3) together with (A_1), implies

$$\begin{aligned} |y_i(t)| &= \beta(t)\Big| - d_ix_i(t) + \sum_{j=1}^n \Big(a_{ij}f_{ij}(x_j(t)) + b_{ij}g_{ij}(x_j(t-\tau_{ij}(t))) + c_{ij}h_{ij}(y_j(t-r_{ij}(t))) \\ &+ \int_{-\infty}^t k_{ij}(t-s)p_{ij}(x_j(s))ds\Big) + l_i(t)\Big| \\ &\leq \beta(t)\Big[d_i|x_i(t)| + \sum_{j=1}^n \Big(|a_{ij}u_{ij}||x_j(t)| + |b_{ij}v_{ij}||x_j(t-\tau_{ij}(t))| + |c_{ij}w_{ij}||y_j(t-r_{ij}(t))| \\ &+ \int_{-\infty}^t |k_{ij}(t-s)\gamma_{ij}||x_j(s)|ds\Big) + L_i\Big], \quad t\in[t_{k-1},t_k), k\in\mathbb{Z}^+, i\in\mathcal{N}. \end{aligned}$$

Again combined with conditions (21) and (22), we have

$$\begin{aligned} 0 &\leq -[y(t)]^+ + \beta(t)\Big(D_0[x(t)]^+ + [A\circ U]^+[x(t)]^+ + [B\circ V]^+[x(t)]^+_\tau + [C\circ W]^+[y(t)]^+_\tau \\ &+ \int_0^{+\infty} [K(s)\circ\Gamma]^+[x(t-s)]^+ds + \bar{L}_y\Big) \\ &\leq \beta(t)\Big(D_0[x(t)]^+ + [A\circ U]^+[x(t)]^+ - \frac{1}{\hat{\beta}}[y(t)]^+ + [B\circ V]^+[x(t)]^+_\tau + [C\circ W]^+[y(t)]^+_\tau \\ &+ \int_0^{+\infty} [K(s)\circ\Gamma]^+[x(t-s)]^+ds + \bar{L}_y\Big), \quad t\in[t_{k-1},t_k), k\in\mathbb{Z}^+. \end{aligned} \tag{29}$$

We note that

$$u = \begin{pmatrix} x \\ y \end{pmatrix} \in C[[t_{k-1},t_k),\mathbb{R}^{2n}]. \tag{30}$$

Let $\mathcal{N}^* = \{1,\dots,2n\}$, $\mathcal{N}_1^* = \mathcal{N}$ and $\mathcal{N}_2^* = \{n+1,\dots,2n\}$. In view of (30) and assumption (A_2), the two inequalities (28) and (29) can be combined into

$$\bar{\Lambda}D^+[u(t)]^+ \leq \beta(t)\left[\bar{P}[u(t)]^+ + \bar{Q}[u(t)]^+_\tau + \int_0^{+\infty} \bar{K}(s)[u(t-s)]^+ds + \bar{L}\right], \tag{31}$$

for $t\in[t_{k-1},t_k)$, $k\in\mathbb{Z}^+$, where

$$\bar{\Lambda} := \mathrm{diag}\{\lambda_1,\cdots,\lambda_{2n}\} \text{ with } \bar{\lambda}_i = 1 > 0 \text{ for } i\in\mathcal{N}_1^* \text{ and } \bar{\lambda}_i = 0 \text{ for } i\in\mathcal{N}_2^*;$$

$$\bar{P} := (\bar{p}_{ij})_{2n\times 2n} \text{ with } \bar{p}_{ij}\geq 0 \text{ for } i\neq j; \bar{Q} := (\bar{q}_{ij})_{2n\times 2n} \text{ with } \bar{q}_{ij}\geq 0 \text{ for } i,j\in\mathcal{N}^*;$$

$$\bar{K}(s) := (\bar{k}_{ij}(s))_{2n\times 2n} \text{ with } \bar{k}_{ij}(s)\geq 0 \text{ and } \bar{k}_{ij}\in L(\sigma_0) \text{ for } i,j\in\mathcal{N}^*; \bar{L}\geq 0.$$

This shows inequality (31) satisfies all conditions (C_1)-(C_3) in Theorem 3.1.

From the initial condition (4), we see that $x_{t_0} = \phi \in PC^1 \subset PC$, $y_{t_0} = \phi' \in PC$. That is

$$u_{t_0} = \bar{\phi} \in PC[(-\infty, 0], \mathbb{R}^{2n}].$$

Noting $\bar{z} > 0$, $\bar{T} \geq 0$, it is easy to deduce

$$\begin{cases} [x(t)]^+ \leq \frac{||\phi||_\infty}{\min\limits_{i\in\mathcal{N}^*}\{\bar{z}_i\}} \bar{z}_x e^{-\sigma \int_{t_0}^{t} \beta(u)du} + \bar{T}_x, & t \in (-\infty, t_0], \\ [y(t)]^+ \leq \frac{||\phi'||_\infty}{\min\limits_{i\in\mathcal{N}^*}\{\bar{z}_i\}} \bar{z}_y e^{-\sigma \int_{t_0}^{t} \beta(u)du} + \bar{T}_y, & t \in (-\infty, t_0], \end{cases}$$

which means

$$[u(t)]^+ \leq \kappa \bar{z} e^{-\sigma \int_{t_0}^{t} \beta(u)du} + \bar{T}, \quad t \in (-\infty, t_0], \tag{32}$$

where $\kappa = \frac{||\phi||_{1\infty}}{\min\limits_{i\in\mathcal{N}^*}\{\bar{z}_i\}} = \frac{||\bar{\phi}||_\infty}{\min\limits_{i\in\mathcal{N}^*}\{\bar{z}_i\}} \geq 0$.

Consequently, under condition (32), applying Theorem 3.1 to inequality (31) for $k = 1$ gives

$$[u(t)]^+ \leq \kappa \bar{z} e^{-\sigma \int_{t_0}^{t} \beta(u)du} + \bar{T}, \quad t \in [t_0, t_1).$$

Suppose that for any $m = 1, 2, \ldots, k$, we have

$$[u(t)]^+ \leq \zeta_0\zeta_1 \cdots \zeta_{m-1} \kappa \bar{z} e^{-\sigma \int_{t_0}^{t} \beta(u)du} + \nu_0\nu_1 \cdots \nu_{m-1}\bar{T}, \quad t \in [t_{m-1}, t_m), \tag{33}$$

with $\zeta_0 = \nu_0 = 1$.

For $t = t_k$, the third equation in (3) together with (A_3) yields

$$[x(t_k)]^+ = [I_k(x(t_k^-))]^+ \leq R_k[x(t_k^-)]^+.$$

We note that conditions (23) and (25) indicate $R_k\bar{z}_x \leq \zeta_k\bar{z}_x$ and $R_k\bar{T}_x \leq \nu_k\bar{T}_x$, respectively. Then from assumption (33) it suffices to obtain

$$\begin{aligned} [x(t_k)]^+ &\leq R_k\left[\zeta_0\zeta_1 \cdots \zeta_{k-1}\kappa\bar{z}_x e^{-\sigma \int_{t_0}^{t_k} \beta(u)du} + \nu_0\nu_1 \cdots \nu_{k-1}\bar{T}_x\right] \\ &\leq \zeta_0\zeta_1 \cdots \zeta_{k-1}\zeta_k\kappa\bar{z}_x e^{-\sigma \int_{t_0}^{t_k} \beta(u)du} + \nu_0\nu_1 \cdots \nu_{k-1}\nu_k\bar{T}_x. \end{aligned} \tag{34}$$

Meanwhile, the fourth equation in (3) together with (A_1) implies that for all $i \in \mathcal{N}$

$$\begin{aligned} |y_i(t_k)| &= |y_i(t_k^+)| = |x_i'(t_k^+)| \\ &= \beta(t_k)\Bigg| - d_ix_i(t_k) + \sum_{j=1}^{n}\Bigg(a_{ij}f_{ij}(x_j(t_k)) + b_{ij}g_{ij}(x_j(t_k - \tau_{ij}(t_k))) \\ &\quad + c_{ij}h_{ij}(y_j(t_k - r_{ij}(t_k))) + \int_{-\infty}^{t_k} k_{ij}(t_k - s)p_{ij}(x_j(s))ds\Bigg) + l_i(t_k)\Bigg| \\ &\leq \beta(t_k)\Bigg[d_i|x_i(t_k)| + \sum_{j=1}^{n}\Bigg(|a_{ij}|u_{ij}|x_j(t_k)| + |b_{ij}|v_{ij}|x_j(t_k - \tau_{ij}(t_k))| \\ &\quad + |c_{ij}|w_{ij}|y_j(t_k - r_{ij}(t_k))| + \int_{-\infty}^{t_k} |k_{ij}(t_k - s)|\gamma_{ij}|x_j(s)|ds\Bigg) + L_i\Bigg]. \end{aligned}$$

Recalling $\zeta_k \geq 1$, $\nu_k \geq 1$ and noting assumption (33) and assertion (34), we have

$$\begin{aligned}
|y_i(t_k)| \quad &\leq \quad \beta(t_k)\Bigg\{d_i\Bigg(\zeta_0\cdots\zeta_k\kappa\bar{z}_i e^{-\sigma\int\limits_{t_0}^{t_k}\beta(u)du} + \nu_0\cdots\nu_k\bar{T}_i\Bigg) \\
&\quad + \sum_{j=1}^{n}\Bigg[|a_{ij}|u_{ij}\Bigg(\zeta_0\cdots\zeta_k\kappa\bar{z}_j e^{-\sigma\int\limits_{t_0}^{t_k}\beta(u)du} + \nu_0\cdots\nu_k\bar{T}_j\Bigg) \\
&\quad + |b_{ij}|v_{ij}\Bigg(\zeta_0\cdots\zeta_k\kappa\bar{z}_j e^{-\sigma\int\limits_{t_0}^{t_k}\beta(u)du} e^{\sigma\hat{\beta}\tau} + \nu_0\cdots\nu_k\bar{T}_j\Bigg) \\
&\quad + |c_{ij}|w_{ij}\Bigg(\zeta_0\cdots\zeta_k\kappa\bar{z}_{j+n} e^{-\sigma\int\limits_{t_0}^{t_k}\beta(u)du} e^{\sigma\hat{\beta}\tau} + \nu_0\cdots\nu_k\bar{T}_{j+n}\Bigg) \\
&\quad + \int\limits_0^{+\infty}|k_{ij}(s)|\gamma_{ij}\Bigg(\zeta_0\cdots\zeta_k\kappa\bar{z}_j e^{-\sigma\int\limits_{t_0}^{t_k}\beta(u)du} e^{\sigma\hat{\beta}s} + \nu_0\cdots\nu_k\bar{T}_j\Bigg)ds\Bigg] + L_i\Bigg\} \\
&\leq \quad \beta(t_k)\Bigg[\zeta_0\cdots\zeta_k\kappa e^{-\sigma\int\limits_{t_0}^{t_k}\beta(u)du}\Bigg(d_i\bar{z}_i + \sum_{j=1}^{n}|a_{ij}|u_{ij}\bar{z}_j + \sum_{j=1}^{n}|b_{ij}|v_{ij}\bar{z}_j e^{\sigma\hat{\beta}\tau} \\
&\quad + \sum_{j=1}^{n}|c_{ij}|w_{ij}\bar{z}_{j+n}e^{\sigma\hat{\beta}\tau} + \sum_{j=1}^{n}\int\limits_0^{+\infty}|k_{ij}(s)|\gamma_{ij}\bar{z}_j e^{\sigma\hat{\beta}s}ds\Bigg) \\
&\quad + \nu_0\cdots\nu_k\Bigg(d_i\bar{T}_i + \sum_{j=1}^{n}|a_{ij}|u_{ij}\bar{T}_j + \sum_{j=1}^{n}|b_{ij}|v_{ij}\bar{T}_j + \sum_{j=1}^{n}|c_{ij}|w_{ij}\bar{T}_{j+n} \\
&\quad + \sum_{j=1}^{n}\int\limits_0^{+\infty}|k_{ij}(s)|\gamma_{ij}\bar{T}_j ds\Bigg) + L_i\Bigg], \quad i \in \mathcal{N}. \qquad (35)
\end{aligned}$$

Together with conditions (21) and (22), we can easily verify inequality (35) has a compact vector form

$$\begin{aligned}
[y(t_k)]^+ \quad &\leq \quad \beta(t_k)\zeta_0\cdots\zeta_k\kappa e^{-\sigma\int\limits_{t_0}^{t_k}\beta(u)du}\Bigg[(D_0 + [A\circ U]^+)\bar{z}_x + [B\circ V]^+e^{\sigma\hat{\beta}\tau}\bar{z}_x \\
&\quad + [C\circ W]^+e^{\sigma\hat{\beta}\tau}\bar{z}_y + \int\limits_0^{+\infty}[K(s)\circ\Gamma]^+e^{\sigma\hat{\beta}s}ds\bar{z}_x\Bigg] + \beta(t_k)\nu_0\cdots\nu_k \\
&\quad \Bigg[(D_0 + [A\circ U]^+)\bar{T}_x + [B\circ V]^+\bar{T}_x + [C\circ W]^+\bar{T}_y \\
&\quad + \int\limits_0^{+\infty}[K(s)\circ\Gamma]^+ds\bar{T}_x + \bar{L}_y\Bigg]. \qquad (36)
\end{aligned}$$

On the other hand, from conditions (18)-(20), we have

$$(D_0 + [A\circ U]^+)\bar{z}_x + [B\circ V]^+e^{\sigma\hat{\beta}\tau}\bar{z}_x + [C\circ W]^+e^{\sigma\hat{\beta}\tau}\bar{z}_y + \int\limits_0^{+\infty}[K(s)\circ\Gamma]^+e^{\sigma\hat{\beta}s}ds\bar{z}_x < \frac{1}{\hat{\beta}}\bar{z}_y. \qquad (37)$$

In addition, $\bar{T} = \bar{D}^{-1}\bar{L}$ is equivalent to $\bar{D}\bar{T} - \bar{L} = 0$, which means $\left(\bar{P} + \bar{Q} + \int\limits_0^{+\infty}\bar{K}(s)ds\right)\bar{T} + \bar{L} = 0$.

In view of conditions (19) and (20), a simple calculation can give

$$(D_0+[A\circ U]^+)\bar{T}_x+[B\circ V]^+\bar{T}_x+[C\circ W]^+\bar{T}_y+\int_0^{+\infty}[K(s)\circ\Gamma]^+ds\bar{T}_x+\bar{L}_y=\frac{1}{\hat{\beta}}T_y. \tag{38}$$

Substituting (37) and (38) into (36), we can deduce

$$[y(t_k)]^+ \quad\leq\quad \zeta_0\cdots\zeta_k\kappa\bar{z}_y e^{-\sigma\int_{t_0}^{t_k}\beta(u)du}+\nu_0\cdots\nu_k\bar{T}_y. \tag{39}$$

Combination of inequalities (34) and (39) gives

$$[u(t_k)]^+\leq\zeta_0\cdots\zeta_k\kappa\bar{z}e^{-\sigma\int_{t_0}^{t_k}\beta(u)du}+\nu_0\cdots\nu_k\bar{T}. \tag{40}$$

Recalling $\zeta_k\geq 1,\nu_k\geq 1$, assumption (33) and assertion (40) show us

$$[u(t)]^+\leq\zeta_0\cdots\zeta_k\kappa\bar{z}e^{-\sigma\int_{t_0}^{t}\beta(u)du}+\nu_0\cdots\nu_k\bar{T},\quad t\in(-\infty,t_k]. \tag{41}$$

Also, it is easy to follow from inequality (31) that

$$\bar{\Lambda}D^+[u(t)]^+\leq\beta(t)\left[\bar{P}[u(t)]^++\bar{Q}[u(t)]^+_\tau+\int_0^{+\infty}\bar{K}(s)[u(t-s)]^+ds+\nu_0\cdots\nu_k\bar{L}\right], \tag{42}$$

for all $t\in[t_k,t_{k+1})$.

Under condition (41), applying Theorem 3.1 again to inequality (42) gives

$$[u(t)]^+\leq\zeta_0\cdots\zeta_k\kappa\bar{z}e^{-\sigma\int_{t_0}^{t}\beta(u)du}+\nu_0\cdots\nu_k\bar{T},\quad t\in[t_k,t_{k+1}).$$

Therefore, it follows from the mathematical induction that

$$[u(t)]^+\leq\zeta_0\cdots\zeta_{k-1}\kappa\bar{z}e^{-\sigma\int_{t_0}^{t}\beta(u)du}+\nu_0\cdots\nu_{k-1}\bar{T},\quad t\in[t_{k-1},t_k),k\in\mathbb{Z}^+. \tag{43}$$

Conditions (24) and (26) imply $\zeta_k\leq e^{\zeta\int_{t_{k-1}}^{t_k}\beta(u)du}$ and $\prod_{m=0}^{k}\nu_m\leq e^{\nu}$ for any $k\in\mathbb{Z}^+$, respectively. This together with (43) yields

$$\begin{aligned}
[u(t)]^+ &\leq e^{\zeta\int_{t_0}^{t_1}\beta(u)du}\cdots e^{\zeta\int_{t_{k-2}}^{t_{k-1}}\beta(u)du}\kappa\bar{z}e^{-\sigma\int_{t_0}^{t}\beta(u)du}+e^{\nu}\bar{T}\\
&= \kappa\bar{z}e^{\zeta\int_{t_0}^{t_{k-1}}\beta(u)du}e^{-\sigma\int_{t_0}^{t}\beta(u)du}+e^{\nu}\bar{T}\\
&\leq \kappa\bar{z}e^{\zeta\int_{t_0}^{t}\beta(u)du}e^{-\sigma\int_{t_0}^{t}\beta(u)du}+e^{\nu}\bar{T}\\
&= \kappa\bar{z}e^{-(\sigma-\zeta)\int_{t_0}^{t}\beta(u)du}+e^{\nu}\bar{T},\quad t\in[t_{k-1},t_k),k\in\mathbb{Z}^+.
\end{aligned}$$

That is

$$[u(t)]^+ \quad\leq\quad \kappa\bar{z}e^{-(\sigma-\zeta)\int_{t_0}^{t}\beta(u)du}+e^{\nu}\bar{T},\ \text{for all } t\geq t_0. \tag{44}$$

Finally, letting $t \to +\infty$ in both sides of (44), we derive

$$\begin{aligned}
\lim_{t\to+\infty}[u(t)]^+ &\leq \kappa\bar{z}\left(\lim_{t\to+\infty} e^{-(\sigma-\zeta)\int\limits_{t_0}^{t}\beta(u)du}\right) + e^{\nu}\bar{T} \\
&= \kappa\bar{z}e^{-(\sigma-\zeta)\lim\limits_{t\to+\infty}\int\limits_{t_0}^{t}\beta(u)du} + e^{\nu}\bar{T} \\
&= e^{\nu}\bar{T}.
\end{aligned}$$

We complete the proof.

Theorem 4.2. Assume (A_1)-(A_3) with $R_k \leq I$ hold. Then

$$\mathcal{A} = \left\{\bar{\phi} \in PC[(-\infty,0],\mathbb{R}^{2n}]\middle|[\bar{\phi}]^+_{\infty} \leq \bar{T}\right\}$$

is a positive invariant set and also a global attracting set of (3).

Proof. By the proof similar to Theorem 4.1, we conclude the key inequality (31) holds. For any initial condition $u_{t_0} = \bar{\phi} \in \mathcal{A}$, we see that

$$[u(t)]^+ \leq \bar{T}, \quad t \in (-\infty, t_0].$$

Under this initial condition, applying Theorem 3.1 with $\kappa = 0$ to (31) for $k = 1$ leads to

$$[u(t)]^+ \leq \bar{T}, \quad t \in [t_0, t_1).$$

For $t = t_1$, $R_k \leq I$ and assumption (A_3) yields

$$[x(t_1)]^+ = [I_1(x(t_1^-))]^+ \leq R_1[x(t_1^-)]^+ \leq [x(t_1^-)]^+ \leq \bar{T}_x.$$

Also, the argument similar to (36), (38) gives

$$\begin{aligned}
[y(t_1)]^+ &\leq \beta(t_1)\Bigg[(D_0 + [A\circ U]^+)\bar{T}_x + [B\circ V]^+\bar{T}_x \\
&\quad + [C\circ W]^+\bar{T}_y + \int\limits_0^{+\infty}[K(s)\circ\Gamma]^+ds\bar{T}_x + \bar{L}_y\Bigg] \\
&\leq \bar{T}_y.
\end{aligned}$$

The above two inequalities means

$$[u(t_1)]^+ \leq \bar{T}.$$

Thus, we obtain

$$[u(t)]^+ \leq \bar{T}, \quad t \in (-\infty, t_1].$$

Clearly, applying Theorem 3.1 with $\kappa = 0$ to the basic inequality (31) for $k = 2$ leads to

$$[u(t)]^+ \leq \bar{T}, \quad t \in [t_1, t_2).$$

Repeating this procedure and by the mathematical induction, we have

$$[u(t)]^+ \leq \bar{T}, \quad t \in [t_{k-1}, t_k), k \in \mathbb{Z}^+.$$

Hence, $\mathcal{A}$ is a positive invariant set of (3).

On the other hand, if $R_k \leq I$, then there must be $\zeta_k = \nu_k = 1$ for $k \in \mathbb{Z}^+$ and $\zeta = \nu = 0$ satisfying assumptions (A_4) and (A_5). It follows from Theorem 4.1 that $\mathcal{A}$ is also a global attracting set of (3).

Remark 4.1. Based on the novel non-autonomous singular delay integro-differential inequality established in Section 3, we investigate the global attracting set and positive invariant set for (3), which have not been considered in [21, 24]. Particularly, if $L_i = 0$ for $i \in \mathcal{N}$, then (A_1) and (A_3) ensure $u^* = 0$ is an equilibrium solution of (3). Furthermore, we have $\bar{T} = \bar{D}^{-1}\bar{L} = 0$, which indicates assumption (A_5) holds and $\mathcal{B} = \{0\}$. In this case, the present Theorem 4.1 shows the following asymptotical stability criterion, which includes Theorem 4.1 in [21] and Theorem 3.2 in [24] as its special cases.

Corollary 4.1. Assume that (A_1)-(A_4) with $L_i = 0$ hold for $i \in \mathcal{N}$. Then the zero solution of (3) is globally asymptotically stable in $PC[(-\infty, 0], \mathbb{R}^{2n}]$. In addition, if $\beta(t) \geq \check{\beta} > 0$, then the zero solution of (3) is globally exponentially stable and the exponential convergence rate is not smaller than $(\sigma - \zeta)\check{\beta}$.

From Remark 2.2, we obtain the following criteria on the global attracting set, positive invariant set and stability for (1).

Theorem 4.3. Assume that (A_1)-(A_5) hold. Then

$$\mathcal{B}' = \left\{ \phi \in PC^1 \middle| [\phi]^+_\infty \leq e^\nu \bar{T}_x, [\phi']^+_\infty \leq e^\nu \bar{T}_y \right\}$$

is a global attracting set of (1).

Theorem 4.4. Assume that (A_1)-(A_3) with $R_k \leq I$ hold. Then

$$\mathcal{A}' = \left\{ \phi \in PC^1 \middle| [\phi]^+_\infty \leq \bar{T}_x, [\phi']^+_\infty \leq \bar{T}_y \right\}$$

is a positive invariant set and also a global attracting set of (1).

Corollary 4.2. Assume that (A_1)-(A_4) with $L_i = 0$ hold for $i \in \mathcal{N}$. Then the zero solution of (1) is globally asymptotically stable in PC^1. In addition, if $\beta(t) \geq \check{\beta} > 0$, then the zero solution of (1) is globally exponentially stable and the exponential convergence rate is not smaller than $(\sigma - \zeta)\check{\beta}$.

Remark 4.2. Particularly, if $\beta(s) \equiv 1$, $p_{ij}(s) \equiv 0$ and $k_{ij}(s) \equiv 0$ for all $s \in \mathbb{R}$ and $i, j \in \mathcal{N}$, then Corollary 4.2 reduces to Theorem 4.2 in [21]. If $\beta(s) \equiv 1$, $f_{ij}(s) \equiv 0$, $v_{ij} = v_j$, $w_{ij} = w_j$, $\gamma_{ij} = \gamma_j$ for all $s \in \mathbb{R}$ and $i, j \in \mathcal{N}$, then Corollary 4.2 reduces to Theorem 3.3 in [24].

Remark 4.3. Corollary 4.2 provides a novel criterion on exponential stability for (1) without requiring the differentiability of delay function τ_{ij} and the monotonicity of f_{ij}, g_{ij} for all $i, j \in \mathcal{N}$, which were required in [22]. Therefore, our method is applicable to a wider range.

5 Illustrative Example

Example. Consider non-autonomous impulsive neutral integro-differential equation with delays

$$\begin{cases} x_1'(t) = (\frac{6}{7} + \frac{1}{7}|\cos(t)|)\Big[-7x_1(t) + \sin(x_1(t))x_1(t - \tau_{11}(t)) - \frac{1}{4}|x_2'(t - r_{12}(t))| \\ \qquad - \int\limits_{-\infty}^{t} e^{-(t-s)}x_1(s)ds + l_1(t)\Big], \quad t \neq t_k, \\ x_2'(t) = (\frac{6}{7} + \frac{1}{7}|\cos(t)|)\Big[-6x_2(t) - |x_2(t - \tau_{21}(t))| + \frac{1}{4}\cos(x_2(t))|x_1'(t - r_{21}(t))| \\ \qquad - \int\limits_{-\infty}^{t} e^{-2(t-s)}x_2(s)ds + l_2(t)\Big], \quad t \neq t_k, \end{cases} \tag{45}$$

with impulsive perturbations

$$\begin{cases} x_1(t_k) = \alpha_{1k}x_1(t_k^-) - \beta_{1k}x_2(t_k^-), \\ x_2(t_k) = \beta_{2k}x_1(t_k^-) + \alpha_{2k}x_2(t_k^-), \end{cases} \tag{46}$$

where α_{ik} and β_{ik} are nonnegative constants, $\tau_{ij}(t) = r_{ij}(t) = |\sin(t)| \leq 1 = \tau$ for $i, j = 1, 2$, the impulsive moments t_k $(k \in \mathbb{Z}^+)$ satisfy: $t_0 = 0 < t_1 < t_2 < \ldots$ and $\lim\limits_{k\to+\infty} t_k = +\infty$.

Clearly,

$$K(s) = (k_{ij}(s))_{2\times 2} \text{ with } k_{11}(s) = e^{-s}, k_{22}(s) = e^{-2s}, k_{12}(s) = k_{21}(s) = 0.$$

Let $\sigma_0 = 0.5$ such that $k_{ij} \in L(\sigma_0)$ for $i, j = 1, 2$.

Denote $\hat{\beta} = 1 \geq \beta(t) = \frac{6}{7} + \frac{1}{7}|\cos(t)|$. The parameters of assumptions (A_1)-(A_3) are as follows

$$D_0 = \begin{pmatrix} 7 & 0 \\ 0 & 6 \end{pmatrix}, \quad [A \circ U]^+ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad [B \circ V]^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$[C \circ W]^+ = \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_k = \begin{pmatrix} \alpha_{1k} & \beta_{1k} \\ \beta_{2k} & \alpha_{2k} \end{pmatrix},$$

$$[K(s) \circ \Gamma]^+ = \begin{pmatrix} e^{-s} & 0 \\ 0 & e^{-2s} \end{pmatrix}, \quad \int_0^{+\infty} [K(s) \circ \Gamma]^+ ds = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

and

$$\bar{P} = \begin{pmatrix} -7 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 7 & 0 & -1 & 0 \\ 0 & 6 & 0 & -1 \end{pmatrix}, \bar{Q} = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{1}{4} & 0 \\ 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{1}{4} & 0 \end{pmatrix}, \int_0^{+\infty} \bar{K}(s) ds = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

A simple calculation by Matlab shows

$$\bar{D} = -\left(\bar{P} + \bar{Q} + \int_0^{+\infty} \bar{K}(s) ds\right) = \begin{pmatrix} 5 & 0 & 0 & -\frac{1}{4} \\ 0 & \frac{9}{2} & -\frac{1}{4} & 0 \\ -9 & 0 & 1 & -\frac{1}{4} \\ 0 & -\frac{15}{2} & -\frac{1}{4} & 1 \end{pmatrix}$$

is an $\mathcal{M}$-matrix. By Lemma 2.1,

$$\Omega_{\mathcal{M}}(\bar{D}) = \left\{ (z_1, z_2, z_3, z_4)^{\mathrm{T}} > 0 \middle| \frac{15}{2} z_2 + \frac{1}{4} z_3 < z_4 < 20 z_1, 9 z_1 + \frac{1}{4} z_4 < z_3 < 18 z_2 \right\}$$

is a nonempty cone without conical surface. It is easy to check that there exist $\bar{z} = (1, 1, 13, 12)^{\mathrm{T}} \in \Omega_{\mathcal{M}}(\bar{D})$ and $\sigma = 0.15 > 0$ satisfying (18). Hence, assumptions (A_1)-(A_3) hold.

Case 1. If $l_1(t) = \cos(5t)$, $l_2(t) = \sin(4t)$, $t_k - t_{k-1} = 2k$ and

$$R_k = \begin{pmatrix} e^{\frac{1}{4^k}} & 0 \\ \frac{1}{4} e^{\frac{1}{4^k}} & \frac{3}{4} e^{\frac{1}{4^k}} \end{pmatrix},$$

then there is $\bar{L}_x = \bar{L}_y = (L_1, L_2)^{\mathrm{T}} = (1, 1)^{\mathrm{T}}$ satisfying $|l_1(t)| \leq L_1$, $|l_2(t)| \leq L_2$ and

$$\bar{T} = \bar{D}^{-1} \bar{L} = (0.6250, 0.7083, 8.75, 8.5)^{\mathrm{T}},$$

which implies $\bar{T}_x = (0.6250, 0.7083)^{\mathrm{T}}$, $\bar{T}_y = (8.75, 8.5)^{\mathrm{T}}$.

Let $\zeta_k = \nu_k = e^{\frac{1}{4^k}}$ for $k \in \mathbb{Z}^+$, $\zeta = \frac{7}{48}$ and $\nu = \frac{1}{3}$. It is easy to verify assumptions (A_4) and (A_5) hold. Thus, by Theorem 4.3,

$$\mathcal{B}' = \left\{ \phi \in PC^1 \middle| [\phi]^+_\infty \leq (0.6250 e^{\frac{1}{3}}, 0.7083 e^{\frac{1}{3}})^{\mathrm{T}}, [\phi']^+_\infty \leq (8.75 e^{\frac{1}{3}}, 8.5 e^{\frac{1}{3}})^{\mathrm{T}} \right\}$$

is a global attracting set of (45) and (46). Fig. 1 shows the simulation result of dynamical behaviors for Case 1.

Remark 5.1. The results in [26, 27] can not be applied to determine the global attracting set for Case 1 because of the neutral term and integral term. The method proposed in [28] is also invalid

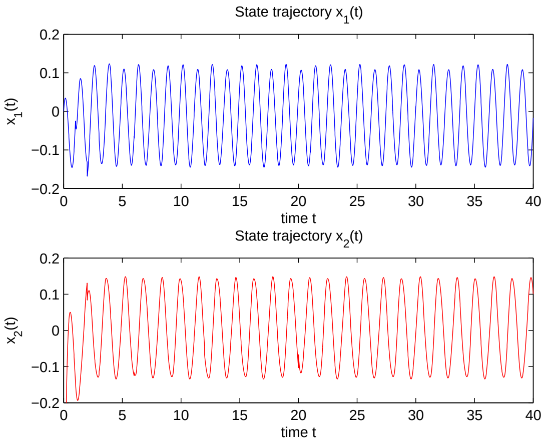

Figure 1: Simulation for Case 1

for Case 1 due to the impulsive perturbations.
Case 2. Consider $l_1(t) = l_2(t) \equiv 0$ for $t \in \mathbb{R}$, $t_k - t_{k-1} = 0.5k$ and

$$R_k = \begin{pmatrix} 0 & e^{0.06k} \\ e^{0.06k} & 0 \end{pmatrix}$$

for $k \in \mathbb{Z}^+$.

Obviously, $L_i = 0$ for $i = 1, 2$. Let $\zeta_k = e^{0.06k}$ for $k \in \mathbb{Z}^+$, $\zeta = 0.14$ and $\check{\beta} = \frac{6}{7}$. A simple calculation implies assumption (A_4) is satisfied. By Corollary 4.2, the zero solution of (45) and (46) is globally exponentially stable. The simulation result is shown in Fig. 2.

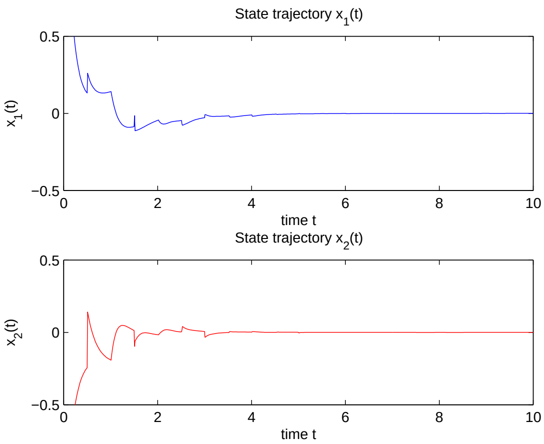

Figure 2: Simulation for Case 2

6 Conclusions

This paper is concerned with the asymptotic behaviors of solutions of a non-autonomous impulsive neutral integro-differential equation with time-varying delays. A novel non-autonomous singular delay integro-differential inequality plays the crucial role in deriving sufficient criteria on the invariant set, global attracting set and exponential stability. The Example illustrates the efficiency of our results. Future work may focus on non-autonomous impulsive stochastic neutral differential equations.

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